Summer Packet for Students Entering
12th Grade Analysis and Approaches HL

Directions:

1. Please complete 3 questions from each topic, one easy (green), one medium (yellow), and one hard (red) for a total of 9 questions. Please do you work on a separate sheet of paper that is clearly and neatly numbered.
2. You must show all of your work for each question.
3. Your teacher will check this assignment on the FIRST day of school at the beginning of class. This will be your first completion grade so make sure you have attempted each problem.
4. If there are questions that you had a difficult time with, please list them in the box below (or highlight them on a separate sheet of paper). We expect you to use the resources provided if you are stuck, but understand there may be additional support needed for some questions.

Assessment:

1. On the second day of math, you will have a summative quiz based on the skills on this summer packet.
2. You will get 5 points for bringing your TI-84 CE Plus to class with you on the day of the test. This is a required tool that you will use throughout high school.

Extra Support:

1. The math department will have extra help days for the summer packet close to the return of school. Please check the school's website during the summer for the dates.
2. On the first day back to school, we will dedicate time in class to go over the answers and for you to ask your teacher questions.

DP Exam Review: Topic 1

1: C	[Maximum mark: 6] Given that $\log_a 2 = 5$. 1. Find the exact value of $\log_a 32$. [2] 2. Find the exact value of $\log_{\sqrt{a}} 2$. [2] 3. Find the value of a, giving your answer correct to 3 significant figures. [2]
2 C	[Maximum mark: 6] On Gary's 50th birthday, he invests $\$P$ in an account that pays a nominal annual interest rate of 5 %, **compounded monthly**. The amount of money in Gary's account **at the end of each year** follows a geometric sequence with common ratio, α. 1. Find the value of α, giving your answer to four significant figures. [3] Gary makes no further deposits or withdrawals from the account. 2. Find the age Gary will be when the amount of money in his account will be **double** the amount he invested. [3]

3
NC

[Maximum mark: 6]

Using mathematical induction, prove that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{Z}^+$.

4
NC

[Maximum mark: 5]

Solve the equation $\log_3(x^2 - 4x + 4) = 1 + \log_3(x - 2)$.

5 NC	[Maximum mark: 6]
	
	Consider the complex number $z = \frac{w_1}{w_2}$ where $w_1 = \sqrt{2} + \sqrt{6}\mathrm{i}$ and $w_2 = 3 + \sqrt{3}\mathrm{i}$.
	
	1. Express w_1 and w_2 in modulus-argument form and write down
	    1. the modulus of z;
	    2. the argument of z.
	2. Find the smallest positive integer value of n such that z^n is a real number.

6 NC	[Maximum mark: 5] The third term of an arithmetic sequence is equal to 7 and the sum of the first 8 terms is 20. Find the common difference and the first term.

7
C

[Maximum mark: 6]

Ten students are to be arranged in a new chemistry lab. The chemistry lab is set out in two rows of five desks as shown in the following diagram.

Desk 1	Desk 2	Desk 3	Desk 4	Desk 5
Desk 6	Desk 7	Desk 8	Desk 9	Desk 10

1. Find the number of ways the ten students may be arranged in the lab. [1]

Two of the students, Hugo and Leo, were noticed to talk to each other during previous lab sessions.

2. Find the number of ways the students may be arranged if Hugo and Leo must sit so that one is directly behind the other. For example, Dest 1 and Desk 6. [2]
3. Find the number of ways the students may be arranged if Hugo and Leo must **not** sit next to each other in the same row. [3]

8
NC

[Maximum mark: 6]

The sum of an infinite geometric sequence is 27. The second term of the sequence is 6. Find the possible values of r.

9
NC

[Maximum mark: 7]

Consider the complex numbers $u = 1 + 2\mathrm{i}$ and $v = 2 + \mathrm{i}$.

1. Given that $\frac{1}{u} + \frac{1}{v} = \frac{6\sqrt{2}}{w}$, express w in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{R}$. [4]
2. Find w^* and express it in the form $re^{\mathrm{i}\theta}$. [3]

10
NC

[Maximum mark: 6]

The 1st, 5th and 13th terms of an arithmetic sequence, with common difference d, $d \neq 0$, are the first three terms of a geometric sequence, with common ratio r, $r \neq 1$. Given that the 1st term of both sequences is 12, find the value of d and the value of r.

11 C	[Maximum mark: 7] Consider the expansion of $\left(2x^6 + \frac{x^2}{q}\right)^{10}$, $q \neq 0$. The coefficient of the term in x^{40} is twelve times the coefficient of the term in x^{36}. Find q.
12 NC	[Maximum mark: 5] Use the fractional binomial theorem to show that $\frac{x}{(1+x)^2} \approx x - 2x^2 + 3x^3$, $\lvert x \rvert < 1$.

13 NC

[Maximum mark: 9]

Consider the following system of equations:

$$x + y + 4z = 1$$
$$3x + 2y + 16z = 5$$
$$4x + 2y + (a-1)z = b-4$$

where $a, b \in \mathbb{R}$.

1. Find conditions on a and b for which
   1. the system has no solutions;
   2. the system has only one solution;
   3. the system has an infinite number of solutions. [6]
2. In the case where the number of solutions is infinite, find the general solution of the system of equations in Cartesian form. [3]

14 C

[Maximum mark: 7]

Given that $(5+nx)^2\left(1+\frac{3}{5}x\right)^n = 25 + 100x + \cdots$, find the value of n.

15 NC	[Maximum mark: 18]

1. Express $-4+4\sqrt{3}\mathrm{i}$ in the form $re^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$.

Let the roots of the equation $z^3=-4+4\sqrt{3}\mathrm{i}$ be z_1, z_2 and z_3.

2. Find z_1, z_2 and z_3 expressing your answers in the form $re^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$.

On an Argand diagram, z_1, z_2 and z_3 are represented by the points A, B and C, respectively.

3. Find the area of the triangle ABC.

4. By considering the sum of the roots z_1, z_2 and z_3, show that

$$\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{4\pi}{9}\right)+\cos\left(\frac{8\pi}{9}\right)=0$$

16 NC	[Maximum mark: 7]

Given that $(1+x)^3(1+px)^4=1+qx+93x^2+\cdots+p^4x^7$, find the possible values of p and q.

17 C	[Maximum mark: 6]

[Maximum mark: 6]

The barcode strings of a new product are created from four letters A, B, C, D and ten digits $0, 1, 2, \ldots, 9$. No three of the letters may be written consecutively in a barcode string. There is no restriction on the order in which the numbers can be written.

Find the number of different barcode strings that can be created.

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18 NC

[Maximum mark: 19]

1. 1. Expand $(\cos\theta + \mathrm{i}\sin\theta)^4$ by using the binomial theorem.

   2. Hence use de Moivre's theorem to prove that

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

   3. State a similar expression for $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Let $z = r(\cos\alpha + \mathrm{i}\sin\alpha)$, where α is measured in degrees, be the solution of $z^4 - \mathrm{i} = 0$ which has the smallest positive argument.

2. Find the modulus and argument of z.

3. Use (a) (ii) and your answer from (b) to show that $8\cos^4\alpha - 8\cos^2\alpha + 1 = 0$.

4. Hence express $\cos 22.5^\circ$ in the form $\dfrac{\sqrt{a+b\sqrt{c}}}{d}$ where $a, b, c, d \in \mathbb{Z}$.

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19
C

[Maximum mark: 15]

Bill takes out a bank loan of $100 000 to buy a premium electric car, at an annual interest rate of 5.49%. The interest is calculated at the end of each year and added to the amount outstanding.

1. Find the amount of money Bill would owe the bank after 10 years. Give your answer to the nearest dollar. [3]

To pay off the loan, Bill makes quarterly deposits of $\$P$ at the end of every quarter in a savings account, paying a nominal annual interest rate of 3.2%. He makes his first deposit at the end of the first quarter after taking out the loan.

2. Show that the total value of Bill's savings after 10 years is $P\left[\frac{1.008^{40}-1}{1.008-1}\right]$. [3]
3. Given that Bill's aim is to own the electric car after 10 years, find the value for P to the nearest dollar. [3]

Melinda visits a different bank and makes a single deposit of $\$Q$, the annual interest rate being 3.5%.

4.
   1. Melinda wishes to withdraw $8000 at the end of each year for a period of n years. Show that an expression for the minimum value of Q is

$$\frac{8000}{1.035}+\frac{8000}{1.035^2}+\frac{8000}{1.035^3}+\cdots+\frac{8000}{1.035^n}.$$

   2. Hence, or otherwise, find the minimum value of Q that would permit Melinda to withdraw annual amounts of $8000 indefinitely. Give your answer to the nearest dollar. [6]

20
NC

[Maximum mark: 17]

1. Solve the equation $z^3 = 27$, $z \in \mathbb{C}$, giving your answer in the form $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and in the form $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$. [6]

2. Consider the complex numbers $z_1 = -1 + \mathrm{i}$ and $z_2 = \frac{1}{\sqrt{2}}\left[\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right]$.
    1. Write z_1 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$.
    2. Calculate $z_1 z_2$ and write in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{R}$.
    3. Hence find the value of $\tan\left(\frac{\pi}{12}\right)$ in the form $c + d\sqrt{3}$ where $c, d \in \mathbb{Z}$.
    4. Find the smallest $p \in \mathbb{Q}^+$ such that $(z_1 z_2)^p$ is a positive real number. [11]

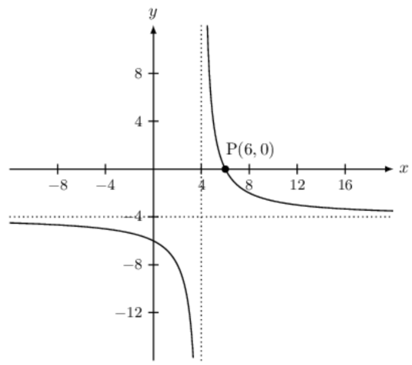
DP Exam Review: Topic 2

1: NC

[Maximum mark: 4]

A rational function is defined by $f(x) = a + \frac{b}{x-c}$, for $x \neq c$, where $a, b, c \in \mathbb{Z}$.

The following diagram represents the graph of $y = f(x)$.



Using the information on the graph,

1. state the value of a and the value of c ; [2]
2. find the value of b. [2]

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2 C

[Maximum mark: 5]

1. Sketch the graph of $y = (x-2)^2 - 4|x-2| - 1$, for $-3 \leq x \leq 7$. [3]
2. Hence, or otherwise, solve the equation $(x-2)^2 - 4|x-2| - 1 = 0$. [2]

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3 C	[Maximum mark: 6] The function f is defined by $f(x) = \sqrt[3]{2x+1}$, for $-14 \leq x \leq 13$. 1. Write down the range of f. [2] 2. Find an expression for f^{-1}. [2] 3. Write down the domain and range of f^{-1}. [2]
4 NC	[Maximum mark: 6] Given that $(x-4)$ is a factor of $f(x) = x^3 - 2x^2 + ax + b$ and that division of $f(x)$ by $(x+2)$ leaves a remainder of 18, find the value of a and the value of b. (ii) $f^{-1}(2)$. [2] (b) Find the domain of f^{-1}. [2] (c) Sketch the graph of $y = f^{-1}(x)$ on the same grid above. [2]

5
NC

[Maximum mark: 5]

The equation $x^2 + (k-3)x - 3k = 0$ has two distinct real solutions.

Find the possible values of k.

6
C

[Maximum mark: 7]

The function f is given by $f(x) = \dfrac{4x^2 - 128}{x^2 - 16}$, for $x \neq \pm 4$.

1. Prove that f is an even function. [2]
2. 1. Sketch the graph of $y = f(x)$.
   2. Write down the range of f. [5]

7
NC

[Maximum mark: 5]

The quadratic equation $x^2 - kx + (k-1) = 0$ has roots α and β. Without solving the equation, find the possible values of the real number k given that $\alpha^2 + \beta^2 = 17$.

8
NC

[Maximum mark: 6]

The function f is defined as $f(x) = \sqrt{\frac{6+2x}{6-2x}}$, for $-3 \leq x < 3$.

Find the inverse function of f, stating its domain and range.

9
C

[Maximum mark: 5]

When $p(x) = x^2 - 2x + p$ is divided by $x - r$, the remainder is 4. Given that $p, r \in \mathbb{R}$, find the largest possible value for p.

10
C

[Maximum mark: 5]

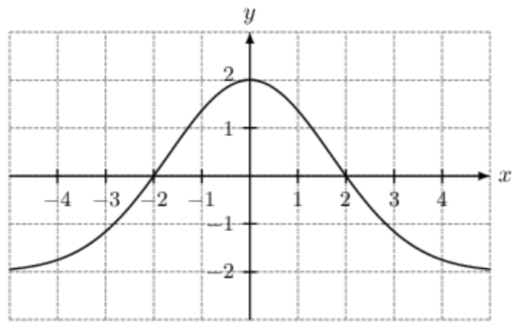
The polynomial $p(x) = 2x^4 + a_3x^3 + a_2x^2 + a_1x - 12$ is divisible by each of $(x+1)$, $(x-1)$ and $(x-2)$.

Find the values of a_1, a_2 and a_3.

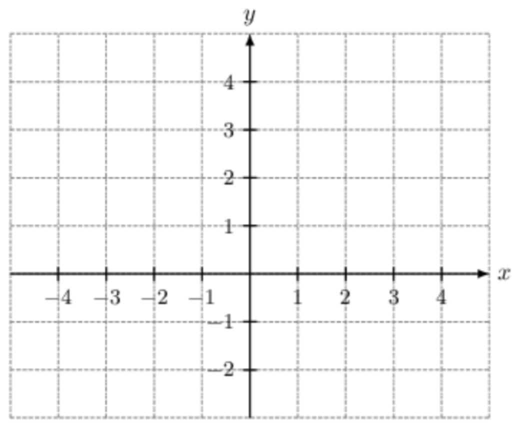
11
NC

[Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -2$. The graph crosses the x-axis at $x = -2$ and $x = 2$, and the y-axis at $y = 2$.



On the following set of axes, sketch the graph of $y = [f(x)]^2 - 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



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12
NC

[Maximum mark: 8]

1. Write down the domain and range of the logarithmic function $y = \log_a x$ where $a > 0$ and $a \neq 1$. [2]

2. Given that $\log_{x^2} y = 9 \log_y(x^2)$, find all the possible expressions of y as a function of x. [6]

13
NC

[Maximum mark: 12]

Consider the polynomial $p(z) = z^5 + z^4 - z^3 + z^2 + 4z + 2$, for $z \in \mathbb{C}$.

1. Write down the sum and product of the roots of $p(z) = 0$. [2]

2. Show that $(z+1)$ is a factor of $p(z)$. [2]

The polynomial can be written in the form $p(z) = (z+1)^3(z^2 + cz + d)$.

3. Find the value of c and the value of d. [5]

4. Hence find the complex roots of $p(z) = 0$. [3]

(c) The equation $(f \circ g)(x) = k$ has exactly two solutions, for $0 \leq x \leq 3$. Find the possible values of k. [3]

14
C

[Maximum mark: 7]

Two distinct roots for the polynomial equation $z^4 - 10z^3 + cz^2 + dz + 170$ are $a + \mathrm{i}$ and $1 + \mathrm{i}b$ where $a, b, c, d \in \mathbb{Z}$, $b > 0$.

1. Write down the other two roots in terms of a and b. [1]
2. Find the value of a and the value of b. [6]

15
C

[Maximum mark: 11]

Consider $f(x) = \frac{1}{2} - \ln\left(\sqrt{x^2 - 4}\right)$.

1. Find the largest possible domain D for f to be a function. [2]

The function f is defined by $f(x) = \frac{1}{2} - \ln\left(\sqrt{x^2 - 4}\right)$, for $x \in D$.

2. Sketch the graph of $y = f(x)$, showing clearly the equations of asymptotes and the coordinates of any intercepts with the axes. [3]
3. Explain why f is an even function. [1]
4. Explain why the inverse function f^{-1} does not exist. [1]

The function g is defined by $g(x) = \frac{1}{2} - \ln\left(\sqrt{x^2 - 4}\right)$, for $x \in (2, \infty)$.

5. Find the inverse function g^{-1} and state its domain. [4]

16
C

[Maximum mark: 12]

Let $f(x) = x^4 - x^3 - 5x^2 + 3x + 2$, for $x \in \mathbb{R}$.

1. Solve the inequality $f(x) < 0$. [3]
2. For the graph of $y = f(x)$, find the coordinates of the local maximum point. Round your answers to **three significant figures**. [3]

The domain of f is now restricted to $[a, b]$ where $a, b \in \mathbb{R}$.

3. 1. Write down the smallest value of $a < 0$ and the largest value of $b > 0$ for which f has an inverse. Give your answers correct to **three significant figures**.
   2. For these values of a and b, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes, showing clearly the coordinates of the end points of each curve.
   3. Solve the equation $f^{-1}(x) = -1$. [6]

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17
C

[Maximum mark: 7]

Let $f(x) = \dfrac{9 - 12x}{cx - 20}$, for $x \neq \dfrac{20}{c}$, where $c \neq 0$.

1. The line $x = 5$ is a vertical asymptote to the graph of $y = f(x)$.
   1. Find the value of c.
   2. Write down the equation of the horizontal asymptote to the graph of $y = f(x)$. [4]
2. The line $y = h$, where $h \in \mathbb{R}$, intersects the graph of $y = |f(x)|$ at exactly one point. Find the possible values of h. [3]

(b) Find $(f \circ f)(1)$. [2]

(c) Sketch the graph of $y = f(-x)$ on the same grid above. [2]

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18
NC

[Maximum mark: 8]

1. Sketch the curve $y = -\left|\dfrac{5}{x-2}\right|$ and line $y = -x - 4$ on the same axes,

   clearly indicating any x and y intercepts and any asymptotes. [3]

2. Find the exact solutions to the equation $x + 4 = \dfrac{5}{|x-2|}$. [5]

19
NC

[Maximum mark: 17]

The function f is defined by $f(x) = 1 + \dfrac{4x}{x+3}$, for $x \neq -3$.

1. Sketch the graph of $y = f(x)$, indicating clearly any asymptotes and points of intersection with the x and y axes. [4]
2. Find an expression for f^{-1}. [4]
3. Find all values of x for which $f(x) = f^{-1}(x)$. [3]
4. Solve the inequality $|f(x)| < 2$. [4]
5. Solve the inequality $f(|x|) < 2$. [2]

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20
C

[Maximum mark: 13]

It is given that $f(x) = 2x^4 + 5x^3 + ax^2 + bx + 4$, for $x \in \mathbb{R}$, where $a, b \in \mathbb{Z}^+$.

1. Given that $x^2 + x - 2$ is a factor of $f(x)$, find the values of a and b. [4]
2. Factorise $f(x)$ into a product of linear factors. [4]
3. Sketch the graph of $y = f(x)$, labeling the maximum and minimum points and the x and y intercepts. [3]
4. Using your graph, state the range of values of c for which $f(x) = c$ has exactly four distinct real roots. [2]

DP Exam Review: Topic 5

1: C	[Maximum mark: 7] A particle moves along a straight line so that its velocity, v ms^{-1}, after t seconds is given by $v(t) = 1.5^t - 4.9$, for $0 \leq t \leq 6$. 1. Find when the particle is at rest. [2] 2. Find the acceleration of the particle when $t = 3$. [2] 3. Find the total distance travelled by the particle. [3]
2 NC	[Maximum mark: 6] Show that $\int_1^3 x^2 \ln x \, dx = 9 \ln 3 - \frac{26}{9}$.

3 NC	

[Maximum mark: 7]

Use L'Hôpital's rule to determine the value of

$$\lim_{x\to 0} \frac{2\sin^2(x)}{\ln(1+x^2)}$$

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4 NC

[Maximum mark: 16]

Let $f(x) = \frac{1}{3}x^3 + 2x^2 - 5x + 10$.

1. Find $f'(x)$. [2]

The graph of f has horizontal tangents at the points where $x = a$ and $x = b$, $a < b$.

2. Find the value of a and the value of b. [3]

3. 1. Sketch the graph of $y = f'(x)$.
   2. Hence explain why the graph of f has a local maximum point at $x = a$. [2]

4. 1. Find $f''(b)$.
   2. Hence, use your answer to part (d) (i) to show that the graph of f has a local minimum point at $x = b$. [4]

The tangent to the graph of f at $x = a$ and the normal to the graph of f at $x = b$ intersect at the point (p, q).

5. Find the value of p and the value of q. [5]

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5
NC

[Maximum mark: 6]

Solve the differential equation

$$(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$$

for y, which satisfies the initial condition $y(0) = -\frac{1}{2}$.

6
NC

[Maximum mark: 17]

Consider a function f. The line L_1 with equation $y = 2x - 1$ is a tangent to the graph of f when $x = 3$.

1. 1. Write down $f'(3)$.
   2. Find $f(3)$. [4]

Let $g(x) = f(x^2 - 1)$ and P be the point on the graph of g where $x = 2$.

2. Show that the graph of g has a gradient of 8 at P. [5]

Let L_2 be the tangent to the graph of g at P. The line L_1 intersects L_2 at the point Q.

3. Find the y-coordinate of Q. [8]

7 C	[Maximum mark: 7] A ladder of length 13 m rests on horizontal ground and leans against a vertical wall. The bottom of the ladder is pulled away from the wall at a constant speed of $1.2\ \mathrm{m\,s^{-1}}$. Calculate the speed of descent of the top of the ladder when the bottom is 5 m away from the wall.
8 NC	[Maximum mark: 11] Farmer Thomas wants to build a sheep farming field in the shape of a rectangle with semicircles of radius r on two sides, as shown on the diagram. He has decided to use in total 350 metres of wooden fencing.  1. 1. Find an expression for the area of the farming field in terms of r. 2. Find the width of the farming field when the area is a maximum. [9] 2. Show that in this case the length of the rectangle is equal to zero and the farming field is the circle of radius $175\pi^{-1}$ metres. [2]

9
NC

[Maximum mark: 6]

Using the substitution $u = e^x - 4$, find $\displaystyle\int \frac{e^x}{e^{2x} - 8e^x + 25}\,\mathrm{d}x$.

10
NC

[Maximum mark: 7]

A curve has equation $2y^2e^{x+1} - 5x^2 = 3$.

1. Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [3]
2. Find the equations of the tangents to this curve when $x = -1$. [4]

11 NC

[Maximum mark: 7]

Consider the functions f, g, defined for $x \in \mathbb{R}$, given by $f(x) = e^{2x} \sin x$ and $g(x) = e^{2x} \cos x$.

1. Find
   1. $f'(x)$;
   2. $g'(x)$. [3]
2. Hence, or otherwise, find $\int_0^{\pi} e^{2x} \cos x \, dx$. [4]

12 C

[Maximum mark: 7]

Let the Maclaurin series for $\cot x$ be

$$\cot x = \frac{a_1}{x} + a_2 x + a_3 x^3 + \cdots$$

where a_1, a_2 and a_3 are non zero constants.

1. Find the series for $\csc^2 x$, in terms of a_1, a_2 and a_3, up to and including the x^2 term
   1. by differentiating the above series for $\cot x$;
   2. by using the relationship $\csc^2 x = 1 + \cot^2 x$. [4]
2. Hence, by comparing your two series, determine the values of a_1, a_2 and a_3. [3]

13 NC

[Maximum mark: 9]

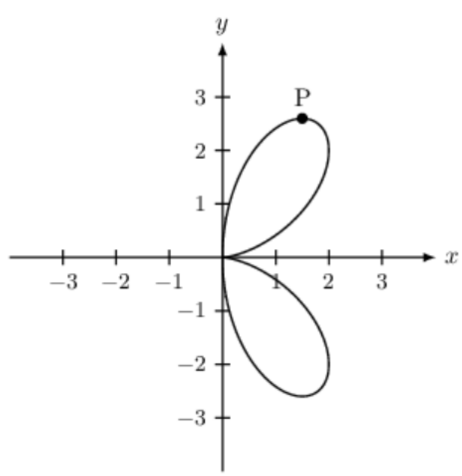
Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{6x}{3x^2 - 2}\right) y = 4x$, given that $y = 4$ when $x = 0$.

1. Show that $3x^2 - 2$ is an integrating factor for this differential equation. [4]
2. Hence solve this differential equation. Give the answer in the form $y = f(x)$. [5]

14 NC

[Maximum mark: 8]

The double folium is a curve defined by the equation $(x^2 + y^2)^2 = 8xy^2$, shown in the diagram below.



Determine the exact coordinates of the point P on the curve where the tangent line is parallel to the x-axis.

15
C

[Maximum mark: 9]

The function f is defined by $f(x) = e^x \cos x$, $x \in \mathbb{R}$.

1. By finding a suitable number of derivatives of f, determine the Maclaurin series for $f(x)$ as far as the term x^4. [6]
2. Hence, or otherwise, determine the exact value of $\lim_{x \to 0} \frac{e^x \cos x - x - 1}{x^3}$. [3]

16
NC

[Maximum mark: 18]

Let $f(x) = \frac{\ln(8x^3)}{kx}$ where $x > 0$, $k \in \mathbb{R}^+$.

1. Show that $f'(x) = \frac{3 - \ln(8x^3)}{kx^2}$. [4]

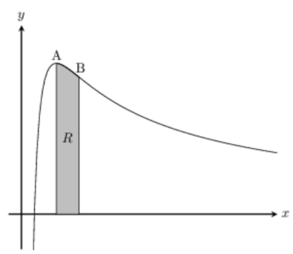
The graph of f has exactly one maximum point A.

2. Find the x-coordinate of A.

The second derivative of f is given by $f''(x) = \frac{2\ln(8x^3) - 9}{kx^3}$. The graph of f has exactly one point of inflexion B.

3. Show that the x-coordinate of B is $\frac{e^{3/2}}{2}$. [3]

The region R is enclosed by the graph of f, the x-axis, and the vertical lines through the maximum point A and the point of inflexion B.



4. Given that the area of R is 5, find the value of k. [8]

17
C

[Maximum mark: 15]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = \frac{1}{2}$, where $x > 0$.

1. Given that $y(1) = 2$, use Euler's method with step length $h = 0.5$ to find an approximation for $y(3)$. Give your answer correct to two significant figures. [6]

2. Solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = \frac{1}{2}$ for $y(1) = 2$. [6]

3. Find the percentage error when $y(3)$ is approximated by the final rounded value found in part (a). Give your answer correct to two significant figures. [3]

18
NC

[Maximum mark: 22]

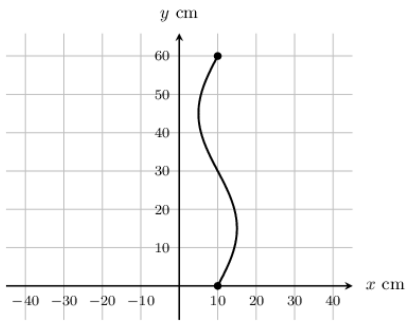
Let $f(x) = (x+1)e^{-2x}$, $x \in \mathbb{R}$.

1. Find $\dfrac{\mathrm{d}f}{\mathrm{d}x}$. [2]
2. Prove by induction that $\dfrac{\mathrm{d}^n f}{\mathrm{d}x^n} = \left[n(-2)^{n-1} + (-2)^n(x+1)\right]e^{-2x}$ for all $n \in \mathbb{Z}^+$. [7]
3. Find the coordinates of any local minimum and maximum points on the graph of $y = f(x)$. Justify whether any such point is a minimum or a maximum. [5]
4. Find the coordinates of any points of inflexion on the graph of $y = f(x)$. Justify whether any such point is a point of inflexion. [5]
5. Hence sketch the graph of $y = f(x)$, indicating clearly the points found in parts (c) and (d) and any intercepts with the axes. [3]

19
NC

[Maximum mark: 19]

The following graph shows the relation $x = 5\sin\left(\frac{\pi y}{30}\right) + 10,\ 0 \le y \le 60$.



The curve is rotated 360° about the y-axis to form a volume of revolution.

1. Calculate the value of the volume generated. [8]

A vase with this shape is made with a solid base of diameter 20 cm. The vase is filled with water from a faucet at a constant rate of 150 $\text{cm}^3\ \text{sec}^{-1}$. At time t sec, the water depth is h cm, $0 \le h \le 60$ and the volume of water in the vase is V cm^3.

2. 1. Given that $\frac{\mathrm{d}V}{\mathrm{d}h} = \pi\left[5\sin\left(\frac{\pi h}{30}\right) + 10\right]^2$, find an expression for $\frac{\mathrm{d}h}{\mathrm{d}t}$.
   2. Find the value of $\frac{\mathrm{d}h}{\mathrm{d}t}$ when $h = 45$ cm. [4]
3. 1. Find $\frac{\mathrm{d}^2h}{\mathrm{d}t^2}$.
   2. Find the values of h for which $\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = 0$.
   3. By making specific reference to the shape of the vase, interpret $\frac{\mathrm{d}h}{\mathrm{d}t}$ at the values of h found in part (c) (ii). [7]

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20
NC

[Maximum mark: 13]

1. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right), \quad x > 0.$$

   Use the substitution $v = \frac{y}{x}$ to show that the general solution of this differential equation is

   $$\int \frac{\mathrm{d}v}{f(v) - v} = \ln x + C.$$ [4]

2. Hence, or otherwise, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x^2 + 5xy + y^2}{x^2}, \quad x > 0,$$

   given that $y = 2$ when $x = 1$. Give your answer in the form $y = g(x)$. [9]